On the Cartan Subalgebra of a Lie Algebra

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ABSTRACT

The task of actually constructing a Cartan subalgebra H of a finite dimensional Lie algebra L over a field F (which is assumed to be p-restricted in case F has characteristic p) leads to a new theorem implying the method of construction, viz.: If X is a nilpotent subalgebra of L of dimension n over F, then any Cartan subalgebra of $L_0(X)=\{u|u\in L \;\&\; \forall x(x\in X \Rightarrow (\mathrm{ad}_L x)^n u=0)\}$ is also a Cartan subalgebra of L.

INTRODUCTION

According to C. Chevalley [2], a subalgebra H of a Lie algebra L over a field F is said to be a *Cartan subalgebra* if it is nilpotent and self-normalizing. Those subalgebras play an important role in the structure theory of L. In this paper we deal with the question of their existence from the constructive viewpoint.

Section 1 contains a preparatory exposition of the constructive aspects of centralizer and normalizer formation.

Section 2 yields two algorithms for the construction of Cartan subalgebras of finite dimensional Lie algebras L over fields of reference F of zero characteristic, which is based on the (apparently new) remark that for any nilpotent F-subalgebra H of L, any Cartan subalgebra of $L_0(H)$ also is Cartan subalgebra of L (Theorem 1). The construction avoids entirely the usual specialization argument of algebraic geometry. In Section 3 the same is done for p-restricted Lie algebras. The case of finite unrestricted Lie algebras remains open.

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Section 4 contains the proofs of the propositions and theorems formulated in Sections 2 and 3.

1. REMARKS ON THE CONSTRUCTION OF CENTRALIZER, NORMALIZER, AND L_0

For any subset X of L, the subset

$$C_L(X) = \{x | x \in L \,\&\, [x, X] = 0\}$$

of all elements of L that annihilate X upon Lie multiplication form an F-subalgebra of L, called the *centralizer* of L. The L-centralizer of 0 is L. The L-centralizer of L is the abelian ideal $C(L)$, the *center* of L. It is contained in every centralizer.

For any F-subalgebra X of L the subset

$$\mathrm{Nor}_L(X) = \{x | x \in L \,\&\, [x, X] \subseteq X\}$$

of all elements of L for which the Lie product of x with any element of X belongs again to X is said to be the *L-normalizer* of X. It is an F-subalgebra of L containing both X and $C_L(X)$ as ideals. If $\mathrm{Nor}_L(X) = X$, then X is said to be a *self-normalizing* F-subalgebra. Such a subalgebra contains the center of L.

For any ideal A of L contained in X the normalizer of X/A in L/A is $\mathrm{Nor}_L(X)/A$:

$$\mathrm{Nor}_{L/A}(X/A) = \mathrm{Nor}_L(X)/A.$$

Thus, if X is self-normalizing in L, then also X/A is self-normalizing in L/A.

For any ordinal number ν the νth *ascending center* of L is defined by transfinite recursion:

$$C_0(L) = L,$$

$$C_1(L) = C(L),$$

$$C_{\nu+1}(L)/C_\nu(L) = C(L/C_\nu(L)),$$

$$C_\nu(L) = \bigcup_{\mu<\nu} C_\mu(L) \qquad \text{if } \nu \text{ is a limit ordinal.}$$

The ascending centers of L form an ascending series of F-ideals of L, the *ascending central series*. Its members are contained in every self-normalizing subring of L.

There is an ordinal number $c(L)$, the *class* of *hypercentrality* of L:

$$C_{c(L)}(L)=C_{c(L)+1}(L),$$
$$C_\nu(L)\subset C_{c(L)}(L) \qquad \text{if} \quad \nu<c(L).$$

The ordinal $c(L)$ is uniquely determined by L. The ideal $C_{c(L)}(L)$ of L is said to be the *hypercenter* of L. It is the union of the members of the ascending central series of L, and it is contained in every self-normalizing subring of L. The center of the factor ring over the hypercenter is 0:

$$C_\nu\big(L/C_{c(L)}(L)\big)=0$$

for all ordinal numbers ν.

Note that the class of hypercentrality is 0 if and only if the center of L is 0. If L is nilpotent, then the class of hypercentrality of L coincides with the class of nilpotency of L.

If L contains a Cartan subalgebra H, then the hypercenter of L is nilpotent and $H/C_{c(L)}(L)$ is a Cartan subalgebra of $L/C_{c(L)}(L)$.

Conversely, if the hypercenter of L is nilpotent, then any Cartan subalgebra $\overline{H}$ of the factor ring of L over its hypercenter defines the Cartan subalgebra of L consisting of the elements of L contained in the cosets of $\overline{H}$ modulo the hypercenter of L.

Thus the task of testing the existence of a Cartan subalgebra of a Lie algebra is reduced to the case that the center is zero.

If the Lie algebra L has a finite bases $b_1,\ldots,b_n$ over F with multiplication rule

$$b_ib_j=\sum\beta_{ij}^kb_k \qquad \big(\beta_{ij}^k\in F;\quad i,j,k=1,2,\ldots,n\big) \tag{1}$$

over F, characterized by the conditions

$$\beta_{ii}^k=0 \qquad (i,k=1,2,\ldots,n) \tag{2a}$$

$$\beta_{ij}^k+\beta_{ji}^k=0 \qquad (1\leqslant i<j\leqslant n,\quad 1\leqslant k\leqslant n) \tag{2b}$$

$$\sum_{k=1}^n\big(\beta_{hi}^k\beta_{kj}^m+\beta_{ij}^k\beta_{kh}^m+\beta_{jh}^k\beta_{ki}^m\big)=0 \qquad (1\leqslant h<i<j\leqslant n,\quad 1\leqslant m\leqslant n), \tag{2c}$$

then the centralizer of a finite subset

$$X = \{x_1, \ldots, x_s\}$$

$$\left(s \in \mathbf{Z}^{>0};\quad x_i = \sum_{k=1}^{n} \xi_i^k b_k,\quad \xi_i^k \in F,\ 1 \leqslant i \leqslant s, 1 \leqslant k \leqslant n \right) \tag{3}$$

is an F-subalgebra $C_L(X)$ with F-basis

$$y_i = \sum_{k=1}^{n} \eta_i^k b_k \qquad \left(1 \leqslant i \leqslant m,\quad m \in \mathbf{Z}^{>0};\quad \eta_i^k \in F,\ 1 \leqslant i \leqslant m, 1 \leqslant k \leqslant n\right), \tag{4a}$$

where the n-rows $(\eta_i^1, \ldots, \eta_i^n)$ $(1 \leqslant i \leqslant m)$ form a solution basis of the linear homogeneous system

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \xi_k^j \eta^i \beta_{ji}^h = 0 \qquad (1 \leqslant h \leqslant s) \tag{4b}$$

for the n-row $(\eta^1, \ldots, \eta^n)$ over F.

In order to determine an F-basis

$$z_i = \sum_{k=1}^{n} \xi_i^k b_k \qquad \left(1 \leqslant i \leqslant m',\ m' \in \mathbf{Z}^{>0};\quad 1_i^k \in F,\ 1 \leqslant i \leqslant m', 1 \leqslant k \leqslant n\right) \tag{5a}$$

of the L-normalizer of a subalgebra given by a finite set of generators (3) over F, we determine a solution basis

$$(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}) \qquad (1 \leqslant i \leqslant m') \tag{5b}$$

of the linear homogeneous system

$$\sum_{k=1}^{n} \lambda_k \xi_j^k = 0 \qquad (1 \leqslant j \leqslant s) \tag{5c}$$

for the n-rows $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ over F and determine the n-rows

$$(\xi_i^1, \xi_i^2, \ldots, \xi_i^n) \qquad (1 \leqslant i \leqslant m')$$

as solution basis of the linear homogeneous system

$$\sum_{k=1}^{n}\sum_{k'=1}^{n}\sum_{k''=1}^{n}\lambda_{ik''}\zeta^{k}\xi_{j}^{k'}\beta_{kk'}^{k''}=0 \qquad (1\leqslant i\leqslant m'',\quad 1\leqslant j\leqslant s) \tag{5d}$$

for the n-rows $(\zeta^1,\zeta^2,\ldots,\zeta^n)$ over F.

In other words, both centralizers and normalizers are constructed by means of linear algebra over fields.

In the same way we construct F-bases of ascending centers $C(L), C_i(L)$ $(1<i)$ as well as bases with corresponding multiplication constants of the factor algebras.

The adjoint representation of L is defined as the F-homomorphism

$$\begin{aligned}&\mathrm{ad}_L: L\to \mathrm{End}_L(L)\\ &\mathrm{ad}_L(x)(u)=[x,u] \qquad (u\in L)\end{aligned} \tag{6}$$

of L into the endomorphism ring of L over F using Lie multiplication by x as action of x on the representation space L. Its kernel is the center of L. In matrix terminology we have

$$\mathrm{ad}_L b_n=\left(\gamma_{hk}^{i}\right), \tag{7a}$$

$$\mathrm{ad}_L\left(\sum_{h=1}^{n}\xi^h b_n\right)=\sum_{h=1}^{n}\xi^h\mathrm{ad}_L b_h, \tag{7b}$$

where $\xi^h\in F$, $1\leqslant h\leqslant n$.

Any nilpotent F-subalgebra H of the Lie algebra L over the field F is contained in the subalgebra $L_0(H)$ of L formed by the elements u of L with the property that for any element of H some power of its adjoint representation annihilates u. The subalgebra $L_0(H)$ of L contains the normalizer of H in L. It coincides with H precisely if H is a Cartan subalgebra of L.

If there is a fixed exponent ν such that

$$(\mathrm{ad}_L h)^{\nu}(u)=0 \qquad [h\in H,\quad u\in L_0(H)], \tag{8}$$

then there holds the *Cartan decomposition*

$$L=L_0(H)\dotplus\hat{H} \tag{9a}$$

of L into the direct sum of $L_0(H)$ and another F-linear subspace $\hat{H}$ subject to the condition

$$[H, \hat{H}] \subseteq \hat{H}. \tag{9b}$$

We note that $\hat{H}$ is uniquely determined as the intersection of the F-linear subspaces $(\mathrm{ad}_L h)^\nu(L)$ ($\nu \in \mathbf{Z}^{>0}$, $h \in H$). Also note that

$$[H, \hat{H}] = \hat{H}. \tag{9c}$$

Of course, if L is finite dimensional, then there is always a Cartan decomposition. Supposing

$$\begin{aligned} H &= \dot{\sum}_{i=1}^{S} Fx_i, \\ L_0(H) &= \dot{\sum}_{i=1}^{S'} Fx_i \qquad (S \subseteq S'), \end{aligned} \tag{9d}$$

then we have

$$\hat{H} = \bigcap_{i=1}^{S} (\mathrm{ad}_L x_i)^{S'}(L) \tag{9e}$$

leading to a construction of $\hat{H}$ in terms of linear algebra over the field of reference.

2. PROPERTIES OF $L_0(H)$

The application of any epimorphism

$$\epsilon : L \to \bar{L} \tag{10a}$$

of a Lie algebra L on the Lie algebra $\bar{L}$ over F to the Cartan decomposition (9a) of L with respect to H yields the Cartan decomposition

$$\bar{L} = L_0(\epsilon H) \dot{+} \epsilon \hat{H} \tag{10b}$$

of $\bar{L}$ with respect to the ϵ-image of H, a nilpotent subalgebra of $\bar{L}$. If H is a Cartan subalgebra of L [i.e. $L_0(H) = H$], then ϵH is a Cartan subalgebra of $\bar{L}$.

In particular, for any ideal A of L and for any Cartan subalgebra H of L, the factor algebra H/A is a Cartan subalgebra of L/A. Trivially, any Cartan subalgebra H of L is a Cartan subalgebra of any subalgebra of L containing H. In particular, H is a Cartan subalgebra of $H+A$ for any ideal of L. Conversely we have

PROPOSITION 1. *Let A be an F-ideal of the finite dimensional Lie algebra L over the field F, and let S be an F-subalgebra of L containing A such that S/A is a Cartan subalgebra of L/A. Then any Cartan subalgebra H of S also is a Cartan subalgebra of L.*

PROPOSITION 2. *For any algebra L and any nilpotent subalgebra H of an ideal A of L, the intersection of $L_0(H)$ with A is the nilspace of $\mathrm{ad}_A H$. If there is a Cartan decomposition of L relative to H, then we have*

$$L_0(H)+A=L. \tag{11}$$

PROPOSITION 3. *If L is a Lie algebra of finite dimension over a field F of zero characteristic, then for a nilpotent Lie algebra H of L over F any Cartan subalgebra X of $L_0(H)$ also is a Cartan subalgebra of L in case $L_0(H)$ is solvable.*

As an application we obtain the following *algorithm for constructing a Cartan subalgebra of a solvable Lie algebra L with a finite number of basis elements $b_1,\dots,b_n$ over a field F of zero characteristic.*

Step 1: Determine $L_0(b_i)$ $(1\leqslant i\leqslant n)$ by means of linear algebra. If $L_0(b_1)=L_0(b_2)=\cdots=L_0(b_n)=L$, then L is the Cartan subalgebra of L. Else there is an index j such that

$$L_0(b_j)\subset L=L_0(b_i) \qquad (1\leqslant i<j);$$

go to step 2.

Step 2: Replace L by $L_0(b_j)$ and go back to step 1.

THEOREM 1. *If L is a finite dimensional Lie algebra over a field F of characteristic zero and H is a nilpotent F-subalgebra of L, then any Cartan subalgebra of $L_0(H)$ also is a Cartan subalgebra of L.*

Using Theorem 1, we propose the following *algorithm for the construction of a Cartan subalgebra* of a Lie algebra L with a finite basis $b_1,\dots,b_n$ over a field F of zero characteristic.

Step 1: If $n \leqslant 1$, then L is the unique Cartan subalgebra of L. If $n > 1$, go to step 2.

Step 2: If $n > 1$, determine $C(L)$. If $C(L) \neq 0$ then go to step 4. Else find an F-basis $x_1, \ldots, x_S$ of a nilpotent F-subalgebra $H^* \neq 0$ of L (e.g. $S = 1$, $x_1 = b_1$, $H^* = Fx_1$ will do). Go to step 3.

Step 3: If $n > 1$, $C(L) = 0$, $H^* = \dot{\sum}_{i=1}^{S} Fx_i$ is a nilpotent F-subalgebra $\neq 0$ of L, then extend $x_1, \ldots, x_S$ to an F-basis $x_1, \ldots, x_{S'}$, of $L_0(H^*)$. If $S' = n$, go to step 5, else apply the algorithm to $L_0(H^*)$ in order to form a Cartan subalgebra H of $L_0(H^*)$. Now H is a Cartan subalgebra of L.

Step 4: If $n > 1$, $C(L) \neq 0$, then extend an F-basis $y_1, \ldots, y_\zeta$ of $C(L)$ to an F-basis $y_1, \ldots, y_n$ of L and form the factor algebra $L/C(L)$ of F-dimension $n - \zeta < n$. Apply the algorithm to $L/C(L)$ in order to extend $y_1, \ldots, y_\zeta$ to the F-basis $y_1, y_2, \ldots, y_{\zeta'}$ of an F-subalgebra H of L containing $C(L)$ such that $H/C(L)$ is a Cartan subalgebra of $L/C(L)$. Now H is a Cartan subalgebra of L.

Step 5: If $n > 1$, $C(L) = 0$, and $H^* = \dot{\sum}_{i=1}^{S} Fx_i$ is a nilpotent F-subalgebra $\neq 0$ of L such that $L_0(H^*) = L = \dot{\sum}_{i=1}^{n} Fx_i$, then L itself is the unique Cartan subalgebra of L in case $S = n$. Else form $C_L(H^*) = \dot{\sum}_{i=1}^{S'} Fx_i$ and go to step 6.

Step 6: If $S' > S$, then replace H^* by $\dot{\sum}_{i=1}^{S+1} Fx_i$ and return to step 3. Else go to step 7.

Step 7: Form $\mathrm{Nor}_L(H^*) = \dot{\sum}_{i=1}^{S''} Fx$. If $S'' = S$, then H^* is a Cartan subalgebra of L. Else go to step 8.

Step 8: If $(\mathrm{ad}_L x_{S+1})^S | H^* = 0$, then replace H^* by $\dot{\sum}_{i=1}^{S+1} Fx_i$ and return to step 3. Else go to step 9.

Step 9: Replace H^* by Fx_{S+1} and return to step 3.

3. THE CASE OF p-RESTRICTED LIE ALGEBRAS

If the characteristic $\chi(F)$ of the field of reference F is not zero then $\chi(F)$ is a prime number p. In this case we use the concept of p-restricted Lie algebras introduced by N. Jacobson [4]. We say that L is *p-restricted* if there is a p-mapping

$$\Pi : L \to L, \qquad \Pi(x) = x^{(p)} \tag{12a}$$

of L in L defined in such a way that

$$\mathrm{ad}_L(x^{(p)}) = (\mathrm{ad}_L x)^p \qquad (x \in L). \tag{12b}$$

In other words, for every element x of L there must be an element y of L such that

$$(\mathrm{ad}_L x)^p = \mathrm{ad}_L y. \tag{12c}$$

The mapping Π is unique precisely if $C(L) = 0$. Otherwise it is unique up to addition of center elements.

For example, the derivation algebra $\mathrm{Der}_F L$ of a Lie algebra L of p over a field F of prime characteristic p is p-restricted, inasmuch as the mapping

$$\begin{gathered} \Pi : \mathrm{Der}_F L \to \mathrm{Der}_F L, \\ \Pi(d) = d^p \qquad [d \in \mathrm{Der}_F L] \end{gathered} \tag{12d}$$

meets the requirement (12b).

In particular, if $C(L) = 0$, then there is the embedding monomorphism

$$\begin{gathered} \mu : L \to \mathrm{Der}_F L \\ \mu(x) = \underline{x} \qquad (x \in L) \\ \mu(x)(u) = [x, u] \qquad (u \in L) \end{gathered} \tag{13a}$$

of L into $\mathrm{Der}_F L$ with the *inner derivation algebra*

$$\mathrm{Inn}\, L = \mu(L) = \{\underline{x} | x \in L\} \tag{13b}$$

as image algebra, an ideal of $\mathrm{Der}_F L$ with the outer derivation algebra

$$\mathrm{Out}_F L = (\mathrm{Der}_F L)/(\mathrm{Inn}\, L) \tag{13c}$$

as cokernel of μ.

In this case L is embedded into the p-restricted subalgebra

$$\dot{\Sigma}_{i=0}^{\infty} F \Pi^i(L) = \bar{L}$$

as the smallest F-subalgebra of L containing L and closed under the application of Π. It contains L as ideal with abelian factor algebra, and of course, it is a p-restricted Lie algebra.

THEOREM 2. *Any finite dimensional Lie algebra over a field F of prime characteristic p contains a Cartan subalgebra, provided L is p-restricted.*

Using Theorem 2, we propose an *algorithm for the construction of a Cartan subalgebra of* L with a finite basis $b_1,\ldots,b_n$ over a field F of prime characteristic p which proceeds along the same lines as the previous algorithm at zero characteristic, only replacing Step 3 by

Step 3′: If $n>1$, $C(L)=0$, $H^*=\dot{\sum}_{i=1}^{S}Fx_i$ is a nilpotent F-subalgebra $\neq 0$ of L, then replace H^* by $\dot{\sum}_{i=1}^{S''}Fx_i = H^* + \dot{\sum}_{i=1}F\Pi(x_j)$. If $S''>S$ repeat. Otherwise extend $x_1,\ldots,x_S$ to an F-basis $x_1,\ldots,x_{S'}$ of $L(H^*)$. If $S'=n$ then go to step 5. Else apply the algorithm to $L_0(H^*)$ in order to form a Cartan subalgebra H of $L_0(H^*)$. Now H is a Cartan subalgebra of L.

4. PROOFS OF THE PROPOSITIONS AND THEOREMS OF SECTIONS 2 AND 3

We precede the proofs by the remark that the concepts centralizer, center, νth center, hypercenter, F-subalgebra, F-ideal, normalizer, self-normalizer, addition and Lie multiplication of F-linear subspaces, nilpotent, Cartan subalgebra, Cartan decomposition, solvable, and p-restricted Lie algebra are invariant under extension of the field of reference. In other words, if L is a Lie algebra over the field F with basis B, and E is a (field) extension of F, then the tensor product algebra $E \underset{F}{\otimes} L = EL$ is a Lie algebra over E with basis B. For any subset X of L we have

$$C_{EL}(X) = E \underset{F}{\otimes} C_L(X).$$

In particular

$$C(EL) = E \underset{F}{\otimes} C(L),$$

and

$$C_\nu(EL) = E \underset{F}{\otimes} C_\nu(L)$$

for any ordinal number ν.

For any F-subalgebra X of L we find $EX = E \underset{F}{\otimes} X$ to be an E-subalgebra of EL such that $\mathrm{Nor}_{EL}(EX) = E \underset{F}{\otimes} \mathrm{Nor}_L(X)$. If X is self-normalizing in L, then

EX is self-normalizing in EL. If X is an F-ideal of L, then EX is an E-ideal of EL.

For any two F-linear subspaces X_1, X_2 of L we have

$$E(X_1 + X_2) = EX_1 + EX_2,$$

$$[EX_1, EX_2] = E[X_1, X_2].$$

If X is a nilpotent F-subalgebra of L, then EX is a nilpotent E-subalgebra of EL of the same class as X.

The nilspace of $\mathrm{ad}_{EL}(EX)$ coincides with $E \underset{F}{\otimes} L_0(X)$. If there is the Cartan decomposition $L = L_0(X) \dot{+} \hat{X}$ of L relative to X then there is the Cartan decomposition

$$EL = EL_0(X) \dot{+} E\hat{X}$$

of EL relative to EX such that

$$\hat{EX} = E \underset{F}{\otimes} \hat{X}.$$

If X is a Cartan subalgebra of L, then EX is a Cartan subalgebra of EL.

If X is an arbitrary F-subalgebra of L, then we find that the derived algebra of EX is given by

$$D(EX) = [EX, EX] = E[X, X] = ED(X),$$

$$D^k(EX) = ED^k(X) \qquad (k \in \mathbf{Z}^{>0}).$$

If X is k-step metabelian, then also EX is k-step metabelian.

Finally, if the characteristic of F is a prime number p and L is a p-restricted Lie algebra, then also EL is p-restricted. This is because every element of EL is of the form

$$x = \sum_{i=1}^{S} \lambda_i \otimes x_i \qquad (s \in \mathbf{Z}^{>0};\ \ \lambda_i \in E, x_i \in L,\ 1 \leqslant i \leqslant S),$$

$$(\mathrm{ad}_{EL} x)^p = \left(\mathrm{ad}_{EL}\left(\sum_{i=1}^{S} \lambda_i \otimes x_i\right)\right)^p = \left(\sum_{i=1}^{S} \lambda_i \mathrm{ad}_{EL} x_i\right)^p.$$

Using the identities developed in [4] (or [6]) we find that

$$(\mathrm{ad}_{EL}x)^p = \sum_{i=1}^{S} \lambda_i^p(\mathrm{ad}_{EL}x_i)^p + \mathrm{ad}_{EL}y$$

for some element y of EL. By assumption we have $(\mathrm{ad}_L x_i)^p = \mathrm{ad}_L(x_i^{(p)})$. Hence

$$(\mathrm{ad}\, X)^p = \sum_{i=1} \lambda_i^p \mathrm{ad}_L\left(x_i^{(p)}\right) + \mathrm{ad}_{EL}y$$

$$= \mathrm{ad}_{EL}z$$

for

$$z = \left(\sum_{i=1}^{S} \lambda_i^p x_i^{(0)}\right) + y.$$

Thus it follows that EL is p-restricted.

Using extension of the field of reference, it suffices to assume that the field of reference F is algebraically closed.

Proof of Proposition 1. There is the Cartan decomposition

$$S = H + \hat{H}$$

of S relative to its Cartan subalgebra. Hence $[H, \hat{H}] = H$, $[H + A, \hat{H} + A] = \hat{H} + A$, and because of the nilpotency of S/A it follows that $\hat{H} \subset A$, $S = H + A$. For any element x of the L-normalizer of H it follows that $[x, H] \subseteq H$, $[x + A, S] \subseteq X$, $[x, S] \subseteq S$, and hence $x \in S$. Since H is a Cartan subalgebra of S it follows that x is in H. Hence H is a Cartan subalgebra of L. ∎

Proof of Proposition 2. Let $L = L(X) + \hat{H}$ be a Cartan decomposition of L relative to H. Since $[H, \hat{H}] = \hat{H}$, $H \subseteq A$, $[A, L] \subseteq A$, it follows that $\hat{H} \subset A$. Hence $A = H + A \cap L_0(H)$, $L(H) + A = L$. ∎

Proof of Proposition 3. Let $L_0(H) = X + \hat{X}$ be the Cartan decomposition of $L_0(H)$ relative to X. By the theorem of Lie, every finite degree irreducible representation Δ of $L(H)$ is of degree 1. Now the equation $[X, \hat{X}] = \hat{X}$ implies that $\Delta\hat{X} = 0$; hence $\Delta(L_0(H)) = \Delta X$.

By assumption, not all irreducible constituents of the restriction of $\mathrm{ad}_L|H$ to $\hat{H}$ are 0. Hence not all irreducible constituents Δ of the restriction of $\mathrm{ad}_L|L_0(H)$ to $\hat{H}$ are 0. Therefore also $\Delta X \neq 0$. It follows that $[X, \hat{H}] = \hat{H}$, $L = X + (\hat{X} + \hat{H})$ is the Cartan decomposition of L relative to X, and X is a Cartan subalgebra of L. ■

In order to prove Theorem 1 for algebraically closed fields F of reference of zero characteristic, we note, first of all, that for any nilpotent F-subalgebra H of the finite dimensional Lie algebra L over the field F there holds the spectral decomposition

$$\hat{H} = \dot{\sum_{\alpha \neq 0}} L_\alpha, \tag{14a}$$

where

$$L_\alpha = \{x | x \in L \ \& \ \forall h [h \in H \Rightarrow (\mathrm{ad}_L h - \alpha)^n(x) = 0]\} \tag{14b}$$

is the eigenspace for the *root* α of H relative to L, and α runs over those nonzero linear forms of H in F for which $L_\alpha \neq 0$.

We note that

$$\alpha([H, H]) = 0 \tag{14c}$$

because of Lie's theorem.

Secondly, we make use of the connected algebraic subgroup $G(L)$ of $\mathrm{End}_F L$ generated by the linear transformations

$$\exp(\mathrm{ad}_L x) = \underline{1}_L + \sum_{i=1}^{n-1} \frac{(\mathrm{ad}_L x)^i}{i!}$$

with x running over the elements of $[h, H]$, L_α $(\alpha \neq 0)$, for all nilpotent subalgebras H of L. It follows again from Lie's theorem that $\mathrm{ad}_L x$ is nilpotent; hence $(\mathrm{ad}_L x)^n = 0$.

By construction the elements of $G(L)$ are automorphisms of L over F. These automorphisms may be called the *inner automorphisms* of L over F. They form a normal subgroup of the group $\mathrm{Aut}_F L$ of all automorphisms of L over F.

The algebraic Lie algebra corresponding to the algebraic group $G(L)$ of linear transformations of L is the ideal $\mathrm{Inn}\, L = \mathrm{ad}_L L$ of $\mathrm{Der}_F L$ which is the algebraic Lie algebra corresponding to $\mathrm{Aut}_F L$ (as tangent space). There holds

PROPOSITION 4. *Any two Cartan subalgebras of L are conjugate under the inner automorphism group of L.*

Proof of Proposition 4. If L is semisimple, then it is well known (see [3]) that $\mathrm{Der}_F L = \mathrm{Inn}\, L$; hence $G(L)$ is the connected component of $\mathrm{Aut}_F L$. Also it is well known from Cartan's structure theory of semisimple finite dimensional Lie algebras over algebraically closed fields of zero characteristic that any two Cartan subalgebras of L are conjugate under $\mathrm{Aut}_F L$.

It remains to show that the Aut L stabilizer of a Cartan subalgebra covers the factor group over Inn L. This follows from the theory of Chevalley groups (see [2]). ∎

By construction any homorphisms

$$\epsilon : L \to L'$$

of a finite dimensional Lie algebra L over F into another finite dimensional Lie algebra L' over F gives rise to the corresponding homomorphism

$$\gamma(\epsilon) : G(L) \to G(L')$$

mapping $\mathrm{ad}_L x$ on $\mathrm{ad}_L i(\epsilon(x))$ for x in $[H, H]$ or in L_α, and $\gamma(\epsilon)$ is surjective (injective) according to the behavior of ϵ.

If L is an arbitrary finite dimensional Lie algebra over F with two Cartan subalgebras H_1, H_2, then $H_i[L, R(L)]$ are Cartan subalgebras of the reductive factor algebra of L over the ideal $[L, R(L)]$ of L (according to Proposition 2). Hence there is an element α of $G[L[R(1)]$ mapping $H_1/[L, R(L)]$ on $H_2/[L, R(L)]$, and there is an element α of $G(L)$ such that α' induces α on $[L, R(L)]$; therefore $\alpha'(H_1)+[L, R(L)] = H_2 + [L, R(L)]$.

Assuming that $H_1 + [L, R(L)] = H_2 + [L, R(L)] = S$, we must prove the existence of an element β of $G[L, R(L)]$ with natural extension to an element β of $G(L)$ mapping H_1 on H_2.

It suffices to deal with the case that L is solvable, and that for some ideal S of L we have $L = H_1 + S = H_2 + S$ and $\mathrm{ad}_L S$ is nilpotent. We want to establish an element β of $G(S)$ naturally extending to an element β' of $G(L)$ such that β' maps H_1 on H_2. This is clear if $S = 0$. Let $S \neq 0$.

Let L be a counterexample of minimal dimension over F. By Lie's theorem there is an ideal A of L of dimension 1 over F contained in S. Hence $H_1/A = (H_i + A)/A$ $(i = 1,2)$ are two Cartan subalgebras of L/A by Proposition 2; by assumption there is an element α of $G(S/A)$ naturally extending to an element α of $G(L/A)$ mapping H_1/A on H_2/A and restricting on S/A to an element of $G(S/A)$. As we saw above, there is an element β of $G(L)$ inducing α' on G/A.

It follows that $\beta'(H_1 + A) = H_2 + A$.

As above, we may assume without loss of generality that $H_1 + S = H_2 + S = L$, $\dim S = 1$, and hence $S \not\subseteq C(L)$. L is nilpotent, and $H_1 = H_2$.

There is no counterexample. Thus Proposition 4 is demonstrated. ■

Proof of Theorem 1. If the theorem is wrong, then there exists a counterexample with minimum value of the F-dimension of L, which for given L assumes the minimum value of the F-dimension of $L_0(H)$; and for given L, $L_0(H)$ assumes the maximum value of $\dim_F H$. Hence there is a Cartan subalgebra S of $L_0(H)$ that is not a Cartan subalgebra of L. It follows that $H \subset L_0(H) \subset L$, $H \neq 0$, $\mathrm{Nor}_L(H) \supset H$, there is a Cartan subalgebra T of $\mathrm{Nor}_L(H)$. Hence T is a Cartan subalgebra of the solvable Lie algebra $H + T$. By Proposition 3 we have $\hat{T} \supseteq \hat{H}$, $L_0(T) \subseteq L_0(H)$. If $L_0(T) = L_0(H)$, then $T \supset H$. This is impossible because of the maximum property of H. Hence $L_0(T) \subset L_0(H)$.

Because of the minimal property of the F-dimension L of $L_0(H)$, it follows that $L_0(T)$ contains a Cartan subalgebra X and that X is a Cartan subalgebra of L. Hence X is a Cartan subalgebra of $L_0(H)$. By Proposition 4 it follows that there is an element α of $G(L_0(H))$ mapping S on X. The natural extension α' of α to an element of $G(L)$ maps S on X. Hence also S is a Cartan subalgebra of L, contrary to assumption.

Thus Theorem 1 is demonstrated ■

The key of the proof of Theorem 2 is contained in the following observation. A linear transformation d of a linear space L over a field F is said to be *algebraic* if it satisfies

$$P(d) = 0 \tag{15a}$$

over a perfect field F where

$$P(t) = \alpha_m t^m + a_{m-1} t^{m-1} + \cdots + \alpha_0 \qquad (m > 0, \quad \alpha_m = 0) \tag{15b}$$

is a nonconstant polynomial in t over F such that

$$\alpha_m d^m + \alpha_{m-1} d^{m-1} + \cdots + \alpha_0 \underline{1}_L = 0. \tag{15c}$$

Now d is the sum of two commuting linear transformations

$$\begin{aligned} d &= d_S + d_n, \\ d_S d_n &= d_n d_S, \end{aligned} \tag{15d}$$

where d_n is nilpotent:

$$d_n^m = 0, \tag{15e}$$

and d_S satisfies a separable equation

$$f(d) = 0 \tag{15f}$$

over F. In other words, f is a polynomial in t over F for which

$$\gcd\left(f, \frac{df}{dt}\right) = 1. \tag{15g}$$

The decomposition (15d) of d into the sum of a *semisimple* component d_S and a *nilpotent component* d_n subject to (15e–g) is unique (see Chevalley [2]).

For example, by divisor cascading of polynomials derived from P by differentiation and, if necessary, by forming the $\chi(F)$th root of a polynomial obtained already, we succeed in factorizing the polynomial P as product of α_m and the power product of monic separable polynomials:

$$P = \alpha_m \prod_{i=1}^{m} P_i^i$$
$$\left[P_i \in F[t];\quad P_i \text{ monic};\quad \gcd\left(P_i, \frac{dP_i}{dt}\right) = 1,\ 1 \leqslant i \leqslant m;\right.$$
$$\left.\gcd(P_i, P_j) = 1,\ 1 \leqslant i < j \leqslant m\right]. \tag{15h}$$

Now we set

$$\bar{P}_i = \begin{Bmatrix} P_i & \text{if} \quad P_i(0) \neq 0 \\ P_i/t & \text{if} \quad P_i(0) = 0 \end{Bmatrix} \qquad (1 \leqslant i \leqslant m), \tag{15i}$$

$$f = \prod_{i=1}^{m} \bar{P}_i, \tag{15j}$$

$$1 = Af + Bt^{i_0} \tag{15k}$$

($i_0 > 0$ if $P_{i_0}(0) = 0$, otherwise $i_0 = 1$; $A, B \in F[t]$),

$$d_S = B(d)d^{i_0}, \qquad d_n = A(d)f(d). \tag{15l}$$

Note that

$$(\lambda d)_S = \lambda d_S, \quad (\lambda d)_n = \lambda d_n \qquad (\lambda \in F). \tag{15m}$$

If E is an extension of the field F then $\underline{1}_E \otimes d$ is a linear transformation of $E \underset{F}{\otimes} L = EL$ such that

$$\begin{aligned} (\underline{1}_E \otimes d)_S &= \underline{1}_E \otimes d_S, \\ (\underline{1}_E \otimes d)_n &= \underline{1}_E \otimes d_n. \end{aligned} \tag{15n}$$

If the field F should happen to be imperfect, then there may not be a decomposition (15d) subject to (15e–g). In that case we can state only that $d^{\chi(F)^\mu}$ is semisimple for any exponent μ satisfying

$$\chi(F)^\mu \geqslant m. \tag{15o}$$

J. P. Serre has observed that d_S, d_n are F-derivations of the F-algebra L in case d is an F-derivation of L (see Serre [5]).

Proposition 5. *Let L be an algebra over the perfect field F. Let H be an F-subalgebra of L. Form the subalgebra H_n of* $\mathrm{Der}_F L$ *generated by the derivations* $(\mathrm{ad}_L h)_n$ *and the subalgebra H_S generated by the derivations* $(\mathrm{ad}_L h)_S$ $(h \in G)$.

It follows that

$$\mathrm{ad}_L H \subseteq H_S + H_n. \tag{16a}$$

If H is nilpotent and the derivations $\mathrm{ad}_L h$ *are algebraic for h of H then*

$$H_S \cap H_n = 0 \tag{16b}$$

$$[H_S, H_n] = 0, \tag{16c}$$

$$[H_S, H_S] = 0, \tag{16d}$$

and if there is a Cartan decomposition of L relative to H, then we have

$$H_S(\hat{H}) = \hat{H}. \tag{16e}$$

Proof of Proposition 5. By the field extension argument it suffices to deal with the case that F is algebraically closed. Then there holds the decomposition

$$L = L_0(H) \oplus \bigoplus_{\alpha \neq 0} L_\alpha \tag{17a}$$

of L into the direct sum of $L_0(H)$ and the root spaces L_α relative to H.

For each element h of H we have

$$h_S(u) = \alpha(h)u \qquad (u \in L_\alpha), \tag{17b}$$

$$h_S(u) = 0 \qquad (u \in L_0(H)), \tag{17c}$$

$$h_n(u) = [\mathrm{ad}_L(h) - \alpha(h)]u \qquad (u \in L_\alpha), \tag{17d}$$

hence H_S is abelian, and also (15b, e) ensue.

Using Levi's theorem as well as the fact that the derivation algebra a semisimple finite dimensional Lie algebra of zero characteristic coincides with the inner derivation algebra, we obtain

COROLLARY 1. *If L is a finite dimensional Lie algebra over a field of zero characteristic, then we have*

$$D(L_S) \subseteq \mathrm{Inn}\, L. \tag{18}$$

Using Proposition 3, we show

COROLLARY 2. *If L is a finite dimensional Lie algebra over a field of zero characteristic, then for any Cartan subalgebra H we find $H_n + H_S$ to be a Cartan subalgebra of $L_S + \mathrm{Inn}\, L$.*

Furthermore we have

COROLLARY 3. *If L is any Lie algebra over a field F of prime characteristic p, then the Lie algebra L^{p^∞} generated by* Inn L *and* Inn L^{p^i} $(i = 1, 2, \ldots)$ *is a subalgebra of* $\mathrm{Der}_F L$ *containing* Inn L *as ideal with abelian factor algebra.*

COROLLARY 4. *If L is a finite dimensional Lie algebra over a field F of prime characteristic p and H is a nilpotent F-subalgebra of L, and if*

$$p^v \geqslant \dim_F L, \tag{19a}$$

then $F(\mathrm{ad}_L H)^{\underline{p}^v}$ *is a p-admissible abelian subalgebra of* $\mathrm{Der}_F L$ *satisfying*

$$(\mathrm{ad}_L H)^{\underline{p}^v}(\hat{H}) = \hat{H}. \tag{19b}$$

Also we have

$$F(\mathrm{ad}_L H)^{\underline{p}^v} = F(\mathrm{ad}_L H)^{\underline{p}^{v'}} \tag{19c}$$

if

$$p^v \geqslant p^{v'} \geqslant \dim_F. \tag{19d}$$

If F is perfect, then we have

$$F(\mathrm{ad}_L H)^{\underline{p}^v} = \sum_{i=0}^{\infty} H_S^{p^i}, \tag{19e}$$

$$\left[(\mathrm{ad}_L H)^{\underline{p}^v}, H_n\right] = 0, \tag{19f}$$

$$\begin{aligned} H^{p^\infty} &= \sum_{i=0}^{\infty} (\mathrm{ad}_L H)^{\underline{p}^i} \\ &= \sum_{i=0}^{\infty} H_{\bar{n}}^{p^i} \oplus F(\mathrm{ad}_L H)^{\underline{p}^v}. \end{aligned} \tag{19g}$$

Now we prove Theorem 2 by using Corollary 4 [especially (19b)].

REFERENCES

1 Roger W. Carter, Simple groups and simple Lie algebras, *J. London Math. Soc.* 40:93–240 (1965).
2 Claude Chevalley, *Théorie des Groupes de Lie, Vol. III*, Hermann, Paris, 1951–55.
3 James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer, New York, 1972.
4 Nathan Jacobson, Restricted Lie algebras of characteristic p, *Trans. Amer. Math. Soc.* 50:15–25 (1941).
5 J. P. Serre, *Lie Algebras and Lie Groups*, Benjamin, New York, 1965.
6 H. Zassenhaus, Über Lie'sche Ringe mit Primzahl charakteristik, *Abh. Math. Sem. Univ. Hamburg* 13:1–100 (1939).
7 H. Zassenhaus, *Lie Groups, Lie Algebras and Representation Theory*, Les Presses de l'Université de Montréal, 1981.

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